Low-frequency dispersion and attenuation in partially saturated rocks

Andrew N. Norris
Rutgers University, Department of Mechanical and Aerospace Engineering, Piscataway, New Jersey 08855-0909

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A theory is developed for the attenuation and dispersion of compressional waves in inhomogeneous fluid-saturated materials. These effects are caused by material inhomogeneity on length scales of the order of centimeters and may be most significant at seismic wave frequencies, i.e., on the order of 100 Hz. The micromechanism involves diffusion of pore fluid between different regions, and is most effective in a partially saturated medium in which liquid can diffuse into regions occupied by gas. The local fluid flow effects can be replaced on the macroscopic scale by an effective viscoelastic medium, and the form of the viscoelastic creep function is illustrated for a compressional wave propagating normal to a layered medium. The wave speeds in the low- and high-frequency limits are associated with conditions of uniform pressure and of uniform "no-flow," respectively. These correspond to the isothermal and isentropic wave speeds in a disordered thermoelastic medium.

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INTRODUCTION

Attenuation of elastic waves in the Earth in the range from zero to several hundred hertz is recognized as having many possible causes, and it is doubtful whether it can be described by any single theory. A summary of some of the effects that have been considered is given by Murphy *et al.*[1] They point out that several processes, such as frictional sliding between grains and internal loss mechanisms in the pore fluid, are not adequate to explain the observed magnitude of attenuation. The theory of Biot[2,3] for fluid saturated porous (permeable) rock is also inadequate if one assumes the porous medium to be spatially uniform.[4-6] Berryman[6] has pointed out, however, that if the permeability is nonuniform then the effective permeability that enters into the attenuation is the average permeability, in contrast to the harmonic average one obtains for steady flow through a heterogeneous rock. Berryman argued that the enhanced permeability could account for seismic wave attenuation within the context of Biot's theory. Other possible mechanisms that have been proposed and studied in detail include local "squirt flow"[1,3] and capillary effects.[7] Jones[8] provided a detailed phenomological description of seismic energy dissipation in terms of linear viscoelastic models.

This paper is concerned with a class of models first proposed by White[9] and developed by White *et al.*,[10] Dutta and Odé,[11,12] and discussed by Dutta and Seriff.[13] The basic premise is that if the medium is only partially saturated with fluid then it is possible for the fluid to undergo "global" motion. The term global is used in contradistinction to local, which indicates motion over lengths on the order of the pore radius; global suggests motion over much greater lengths. The fluid motion is associated with pore pressure diffusion and can result in relatively large attenuation. The frequency range is such that the diffusion length is commensurate with the scale of inhomogeneity associated with the partial saturation, and both lengths are much less than the wavelength. The precise nature of the scaling is discussed in the next section. The general mechanism does not assume partial saturation, but only that the medium is inhomogeneous. For example, the pores could be completely saturated with liquid but the compressibility of the solid frame may vary with position. However, the diffusion is greatest if the fluid compressibility varies significantly from point to point. Thus, White[9,10] and others have concentrated on rocks infiltrated by both gas and liquid. The gas and liquid are assumed to occupy distinct parts of the same connected pore space so that the liquid may easily diffuse into the region occupied by gas.

In his original study White[9] considered a spherical region of gas or liquid surrounded by a concentric shell of liquid or gas. The macroscopic attenuation of seismic energy was assumed to be related to an effective modulus which is determined by subjecting the representative sample to oscillatory forcing and estimating the bulk response, i.e., the average dilatation in the spherical model of White.[9] The resulting effective modulus is complex, implying dispersion and attenuation. White *et al.*,[10] using the same procedure as White,[9] considered a layered medium with alternating liquid and gas zones. Subsequently, Dutta and Odé[11,12] considered the same spherical model as White,[9] but they solved the microproblem by modeling each region using the more sophisticated Biot theory for fluid saturated media. The pore pressure diffusion is then replaced by the Biot second wave, or slow wave, which reduces to a diffusive process at low frequency. The analysis of Dutta and Odé was necessarily quite a bit more complicated than White's, but they found that the predicted attenuation and dispersion were not appreciably different. Dutta and Odé also noted that the attenuation is not strongly affected by the microstructure (spherical versus layer models).

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The purpose of this paper is to develop a macroscopic theory which takes into account the type of microstructure in the White model. More generally, the theory developed here allows us to "average" material inhomogeneity occurring on length scales commensurate with the pore pressure diffusion length. First, it is shown by a detailed study of the specific model of a periodically layered poroelastic medium that the effective wave number of compressional waves is defined by the solution to a problem equivalent to that studied by White *et al.*[10] This result provides a rigorous basis for the concept of a frequency-dependent effective modulus, which was not properly justified by previous authors[9–13] in terms of a specific, complete model. In this paper the microstructural mechanics is assumed to be described by the Biot theory for fluid-filled media. By allowing the Biot continuum to be inhomogeneous and applying ideas from homogenization theory, it is shown that the macroscopic effective medium is simply viscoelastic. Thus, the Biot equations, which are themselves homogenized versions of the coupled equations of elasticity and fluid dynamics,[14] are homogenized even further in this paper. Some general results are also derived for the limiting low- and high-frequency moduli of the effective viscoelastic medium, for both the simple layered model and in general for full three-dimensional heterogeneity. It is shown by analogy that these moduli are directly related to the isothermal and isentropic moduli of disordered thermoelastic materials, and that the analogy is of practical use in obtaining the moduli. Explicit expressions are given for the limiting moduli for different situations, including the full set of transversely isotropic moduli in a layered medium.

The layout of the paper is as follows. First, the physical scalings involved are defined and discussed, without reference to any specific problem. Biot's poroelasticity equations are then applied to the particular problem of compressional waves propagating through a periodically layered medium. This problem is chosen because it is probably the simplest for which the White model can be deduced from first principles, with explicit expressions for many of the effective parameters. For instance, it is shown that the effective modulus corresponds to a causal viscoelastic creep function, which is illustrated for the case of a water/gas-saturated layered medium. The connection between the layered medium problem and the two-phase model of White *et al.*[10] is shown explicitly in Appendix A. The remainder of the paper deals with the limiting moduli for arbitrary, three-dimensional heterogeneity, in the limits of low and high frequency. The disparity in the limiting moduli indicates the strength of the attenuation and dispersion, and despite the fact that they cannot be easily obtained, it is possible to make some general but meaningful statements about the values of these moduli using a simple analogy with thermoelasticity.

I. SCALING

There are many lengths associated with wave propagation in fluid-saturated rock, ranging from the total distance of propagation which may be on the order of a kilometer, to the shortest characteristic length scales of the microstructure in the rock, which is certainly on the submicron level. It is helpful to focus on four particular lengths, some better defined than others, but each quite distinct: The wavelength, the pore radius, the viscous skin depth, and the diffusion length,

$$\lambda = v/f, \quad L_p = (8F\kappa)^{1/2}, \quad L_v = \left(\frac{2\eta}{\omega\rho_f}\right)^{1/2}, \quad L_d = \left(\frac{D_0}{\omega}\right)^{1/2}, \tag{1}$$

respectively, where v is the wave speed, f is the frequency, and the remaining parameters will be defined presently. In order to appreciate the relative magnitudes, it is instructive to consider compressional waves in water-saturated Berea sandstone (see Tables I and II) at the specific frequency $f = 100$ Hz, for which $\lambda = 36.7$ m. A much shorter length scale can be defined from the value of the permeability κ, which is a measure of area. The square root of κ gives an estimate of the typical pore radius, or more accurately, the average pore dimension in rocks can be defined as L_p (D. L. Johnson, private communication), where F is the formation factor of the rock.[15] An empirical rule of thumb for rocks is that $F \approx 1/\phi^2$, where ϕ $(0<\phi<1)$ is the porosity, implying $L_p = 6.66$ μm for the case considered. Both L_v and L_d are intermediate length scales associated with diffusion. The viscous skin depth depends upon $\omega = 2\pi f$, the circular frequency, η the fluid viscosity, and ρ_f the fluid density. The skin depth for water saturation at standard temperature and pressure is 56 μm, which exceeds the pore radius by an order of magnitude, indicating that fluid mobility is low in the pore space and is dominated by the viscous forces causing the fluid to adhere to the pore walls. A critical frequency, $\omega_c = 2\pi f_c$, exists above which the fluid is freed from its viscous ties and can undergo relatively large motion with respect to the frame (within the limits of infinitesimal linear theory!). It may be defined as the frequency at which $L_p = L_v$, implying $\omega_c = \eta/4F\rho_f\kappa$. Thus, $f_c = 7.2$ kHz for the example considered. The final length scale L_d is defined by pore fluid diffusion within the solid, and depends upon the diffusion coefficient $D_0 = \kappa K_f/\eta\phi$,[16,17] where K_f is the fluid bulk modulus. Alternatively,

TABLE I. Data for Berea sandstone. The shear modulus is $\mu = (1-\phi)\rho_s\bar{v}_s^2$ and the bulk modulus is $K = (1-\phi)\rho_s\bar{v}_p^2 - 4\mu/3$. The underlying grain is presumed to have bulk modulus $K_g = 3.79\times10^{11}$ g/cm s^2.

Parameter	Symbol	Value	Units
Porosity	ϕ	0.19	...
Permeability	κ	0.2	darcy $=10^{-8}$ cm^2
Compressional speed	$\bar{v}_p$	3670	m/s
Shear speed	$\bar{v}_s$	2170	m/s
Density	ρ_s	2.65	g/cm^3

TABLE II. The fluid parameters used in the numerical calculations. The fluid bulk modulus is $K_f = \rho_f v_f^2$.

Fluid	η (g/cm s)	ρ_f (g/cm^3)	v_f (m/s)
Water	1.0×10^{-2}	1.00	1500
Gas	2.2×10^{-4}	0.14	630

TABLE III. The four lengths for water-saturated Berea sandstone at 100 Hz.

Length	Symbol	Value (m)
Wavelength	λ	3.7×10^{1}
Pore pressure diffusion length	L_d	6.1×10^{-2}
Viscous skin depth	L_v	5.6×10^{-5}
Pore radius	L_p	6.7×10^{-7}

the identity $K_f=\rho_f v_f^2$, where v_f is the fluid sound speed, implies $D_0=(4\phi F\omega_c)^{-1}v_f^2$. The diffusion length $L_d=6.1$ cm in the example considered. The four lengths are tabulated in Table III from which it is evident that they are quite disparate, with no two of similar magnitude.

The main purpose of this paper is to examine the effect of material inhomogeneity on the length scale of the pore pressure diffusion length L_d. Let L be a typical length over which the material properties vary. For instance, L may be the length of a unit period in a periodically layered medium, or more generally, L may be the spatial autocorrelation length in a nonperiodically layered medium. The following scaling is assumed:

$$\lambda \gg L, \quad L=O(L_d), \quad L\gg L_v \gg L_p. \tag{2}$$

The final inequality implies that the quasistatic approximation to the Biot theory for saturated rock is adequate. If $L_p \gg L_v$, then in principle, the diffusion of pore pressure could be replaced by the Biot second wave.[2] This possibility is examined by Dutta and Odé[11,12] who generalized the spherical White model using the full Biot theory.

We are here concerned with situations where the heterogeneity of the porous medium results from the presence of both liquid and gas in the pore space, each fluid having quite different mechanical properties. The medium may then be thought of as partially saturated in the sense that the liquid is the dominant saturant. This situation is distinct from that considered by Berryman and Thigpen[18] who generalized the Biot theory to account for the possibility of more than one fluid in the pores. Berryman and Thigpen assumed that each fluid is present at every material point, and therefore they introduced new variables for the additional fluids. It is assumed here that there is only a single fluid at any given point, but the fluid at neighboring points, on the order of L distant, may be different. The present theory considers an inhomogeneous porous medium in the sense of Biot[3] with L as the length scale, whereas the theories of Berryman and Thigpen[18] and others for partially saturated porous media attempt to derive macroscopically homogeneous equations that incorporate inhomogeneity on the length scale L_p.

II. WAVES IN A HORIZONTALLY LAYERED MEDIUM

A. The system of equations

The theory of dynamic poroelasticity as first derived by Biot[2] has been discussed extensively in the literature; for example, Burridge and Keller[14] derived the equations of motion from first principles using homogenization theory. The notation of Biot's 1962 paper[3] is used here. The solid (matrix) and relative fluid displacements are $\mathbf{u}$ and $\mathbf{w}=\phi(\mathbf{u}_f-\mathbf{u})$, where $\mathbf{u}_f$ is the total fluid displacement. The bulk stress tensor τ and the pore pressure p are

$$\tau=\mathbf{L}_c\mathbf{e}-\alpha M\xi\mathbf{I}, \quad p=-\alpha Me+M\xi, \tag{3}$$

where $\mathbf{e}$ is the solid strain, $e=\text{div}\,\mathbf{u}$, $\xi=-\text{div}\,\mathbf{w}$, and $\mathbf{I}$ is the second-order identity tensor. The isotropic confined stiffness tensor $\mathbf{L}_c$ involves the bulk and shear moduli K_c and μ. The corresponding unconfined bulk modulus is $K<K_c$. Some of the moduli and parameters are related, e.g.,[3]

$$K_c=K+\alpha^2 M, \quad \frac{1}{M}=\frac{\phi}{K_f}+\frac{(1-\alpha)(\alpha-\phi)}{K}, \tag{4}$$

where K_f is the fluid bulk modulus. The parameter α, $\phi<\alpha<1$, can be related to the bulk modulus of the granular material K_g, by $\alpha=1-K/K_g$.[19]

We consider time harmonic longitudinal wave motion traveling normal to the layers in a layered medium, with $e^{-i\omega t}$ understood but omitted. The material parameters in the Biot equations are functions of a single coordinate only, say z, and the displacement vectors $\mathbf{u}$ and $\mathbf{w}$ are polarized in the z direction, with components u and w. The natural or open-pore boundary conditions of poroelasticity require that the four-vector $\mathbf{V}=(\dot{u},\dot{w},-\tau_{zz},p)^T$ is continuous for all z, even if the material properties are discontinuous. The equations of motion[3] and the constitutive relations (3) can be reduced to a system of first order equations for $\mathbf{V}$,

$$\frac{d}{dz}\mathbf{V}=i\omega\begin{bmatrix}0 & \mathbf{S}\\ \mathbf{R} & 0\end{bmatrix}\mathbf{V}(z)-\mathbf{E}(z)\mathbf{V}(z), \tag{5}$$

where

$$\mathbf{S}=\begin{bmatrix}\dfrac{1}{K+4/3\mu} & \dfrac{-\alpha}{K+4/3\mu}\\ \dfrac{-\alpha}{K+4/3\mu} & \dfrac{K_c+4/3\mu}{M(K+4/3\mu)}\end{bmatrix}, \quad \mathbf{R}=\begin{bmatrix}\rho & \rho_f\\ \rho_f & \tilde{m}\end{bmatrix}. \tag{6}$$

Here, $\rho=\phi\rho_f+(1-\phi)\rho_s$, ρ_s, and ρ_f are the average, grain, and fluid densities, respectively. The only nonzero element of $\mathbf{E}$ is $E_{42}=\eta/\kappa$, which together with the $\tilde{m}$ element of $\mathbf{R}$ defines the viscodynamic operator of the pore space, $\eta/\kappa+(-i\omega)\tilde{m}(\omega)$, where η is the viscosity and κ the static permeability. The dynamic permeability effects are lumped into the frequency-dependent density $\tilde{m}$. Our results will not depend upon the latter quantity, so we say no more about it here but refer to Norris[20] for further details. The form of $\mathbf{E}$ in Eq. (5) emphasizes the dominant role of the Darcy flow term at low frequency, resulting in dissipation in the homogenized theory derived below.

Let ρ_0 and v_0 be typical average values of density and compressional wave speed, and L some length, which can be identified with the length L of the Sec. I. Let ω_{c0} be a typical value of the critical frequency ω_c defined in Sec. I. Define the dimensionless parameters

$$\epsilon=\frac{v_0}{\omega_{c0}L}, \quad X=\frac{z}{L}, \quad \Omega=\frac{\omega L}{v_0}\frac{\omega_{c0}L}{v_0}, \tag{7}$$

and put

$$\dot{u}=\frac{\bar{u}}{\rho_0 v_0},\quad \dot{w}=\epsilon\frac{\bar{w}}{\rho_0 v_0},\tag{8}$$

where $\bar{u}$ and $\bar{w}$ both have the dimensions of stress. Then Eq. (5) becomes

$$\begin{aligned}\frac{d\bar{u}}{dX}&=-i\Omega\epsilon(\bar{S}_{11}\tau_{zz}+\bar{S}_{12}p),\\ \frac{d\tau_{zz}}{dX}&=-i\Omega\epsilon(\bar{R}_{11}\bar{u}-\epsilon\bar{R}_{12}\bar{w}),\\ \frac{d\bar{w}}{dX}&=i\Omega(\bar{S}_{12}\tau_{zz}+\bar{S}_{22}p),\\ \frac{dp}{dX}&=-i\Omega\epsilon(\bar{R}_{12}\bar{u}-\epsilon\bar{R}_{22}\bar{w})-a\bar{w},\end{aligned}\tag{9}$$

where $\bar{\mathbf{S}}=\rho_0 v_0^2\mathbf{S}$, and $\mathbf{R}=\mathbf{R}/\rho_0$, are dimensionless compliance and mass density matrices, and

$$a(X)=\frac{\eta}{\kappa}\frac{1}{\rho_0\omega_{c0}}.\tag{10}$$

B. Decoupling of the pore pressure

It is assumed that $\epsilon\ll1$, and that the four variables $\bar{u}$, τ_{zz}, $\bar{w}$, and p, all of which have dimensions of stress, are of the same order of magnitude. Then retaining only the leading order terms in the right members of Eq. (9) gives

$$\begin{aligned}\frac{d\bar{u}}{dX}&=-i\Omega\epsilon(\bar{S}_{11}\tau_{zz}+\bar{S}_{12}p),\\ \frac{d\tau_{zz}}{dX}&=-i\Omega\epsilon\bar{R}_{11}\bar{u}\end{aligned}\tag{11}$$

and

$$\begin{aligned}\frac{d\bar{w}}{dX}&=i\Omega(\bar{S}_{12}\tau_{zz}+\bar{S}_{22}p),\\ \frac{dp}{dX}&=-a\bar{w}.\end{aligned}\tag{12}$$

The bulk variables $\bar{u}$ and τ_{zz} vary slowly because of the presence of the factor ϵ in their differential equations, whereas the pore variables $\bar{w}$ and p vary over length scales $X=O(1)$. The bulk stress τ_{zz} can therefore be approximated as constant, correct to $O(\epsilon)$, in the equations for $\bar{w}$ and p, and $(\bar{w},p)$ can be solved for, subject to some boundary conditions discussed below. The pressure can then be put back into the "macroscopic" equation for $\bar{u}$ and τ_{zz}, forming a closed system. Equations (11) and (12), although not rigorous, indicate how the pore diffusion effects decouple from the bulk wave propagation, provided $\epsilon\ll1$, or $v_0\ll\omega_{c0}L$, see Eq. (7). These rather vague but physically reasonable concepts will now be made more precise by considering the asymptotic approximation (in ϵ) to the exact equations in a periodically stratified medium. It will become clear that the preceding approximate analysis is rigorously justified in an asymptotic sense.

C. Periodic stratification

The period is L, or unity in terms of X of Eq. (7), and averages of a quantity f over a unit period are denoted by $\langle f\rangle$. Define ρ_0 as $\langle\rho\rangle$, and $v_0=(C_\infty/\rho_0)^{1/2}$, where

$$C_\infty=\langle 1/(K_c+\tfrac{4}{3}\mu)\rangle^{-1}\tag{13}$$

is the effective axial stiffness of the equivalent transversely isotropic homogenized continuum. The suffix ∞ applies to the high-frequency limit, $\Omega\gg1$. It will be shown that v_0 is the limiting speed of compressional waves in the high-frequency limit. "High frequency" means that the diffusion length is far less than the inhomogeneity scale, although the wavelength is still assumed to be much longer than any other characteristic length and the frequency much less than the reference critical frequency ω_{c0}, appearing in Eq. (7), which can be taken as the average of ω_c through the unit period. The exact choice of ρ_0, v_0, and ω_{c0} is arbitrary, and the final results should not depend upon them, but in principle they should be such that $\Omega=O(1)$ and $\epsilon\ll1$. This allows the use of formal asymptotic methods to find the leading order asymptotic approximation in ϵ to the compressional dispersion relation.

The exact solution can be represented using Bloch waves, which are solutions of the form

$$\bar{V}(X)=\bar{\mathrm{U}}(X)e^{i\bar{k}X},\tag{14}$$

where $\bar{\mathrm{U}}$ is periodic in X, with unit period. There is a countably infinite set of Bloch modes, each with its own dispersion relation, $\bar{k}=\bar{k}(\omega)$, for the complex-valued wave number. The bulk compressional wave corresponds to the lowest or fundamental branch, suggesting the ansatz

$$\bar{k}=\epsilon\Omega\bar{k}_0+(\epsilon\Omega)^2\bar{k}_1+\cdots.\tag{15}$$

The periodicity condition for the mode is, from Eqs. (14) and (15),

$$\bar{V}(1)=\bar{V}(0)[1+i\epsilon\Omega\bar{k}_0+O(\epsilon^2)].\tag{16}$$

At the same time, Eqs. $(9)_1$ and $(9)_2$ can be integrated to give for $0<X\leqslant1$,

$$\begin{aligned}\bar{u}(X)&=\bar{u}(0)-i\epsilon\Omega\int_0^X(\bar{S}_{11}\tau_{zz}+\bar{S}_{12}p)dX',\\ \tau_{zz}(X)&=\tau_{zz}(0)-i\epsilon\Omega\int_0^X\bar{R}_{11}\bar{u}\,dX'+O(\epsilon^2).\end{aligned}\tag{17}$$

These imply that the bulk fields u and τ_{zz} are constant to first order in ϵ throughout the layer, and so Eq. (17) can be iterated to obtain

$$\begin{aligned}\bar{u}(X)&=\bar{u}(0)-i\epsilon\Omega\tau_{zz}(0)\int_0^X\bar{S}_{11}\,dX'\\ &\quad-i\epsilon\Omega\int_0^X\bar{S}_{12}p\,dX'+O(\epsilon^2),\\ \tau_{zz}(X)&=\tau_{zz}(0)-i\epsilon\Omega\bar{u}(0)\int_0^X\bar{R}_{11}\,dX'+O(\epsilon^2).\end{aligned}\tag{18}$$

The other pair of equations in (9) then yields, with the same approximations,

$$\frac{d\bar{w}}{dX}=i\Omega[\bar{S}_{12}\tau_{zz}(0)+\bar{S}_{22}p]+O(\epsilon),$$
$$\frac{dp}{dX}=-a\bar{w}+O(\epsilon). \tag{19}$$

These are a pair of forced linear equations for $\bar{w}$ and p driven by the macroscopic stress $\tau_{zz}(0)$; the decoupling is now complete. The solution for p can be written as

$$p(X)=q(X)\tau_{zz}(0)+O(\epsilon), \tag{20}$$

where $q(X)$ is independent of $\tau_{zz}(0)$, and when this is substituted back into Eq. (18) explicit expressions are obtained for $\bar{u}$ and τ_{zz}, correct to $O(\epsilon)$. Putting $X=1$ yields

$$\begin{pmatrix} -\bar{u}(1) \\ \tau_{zz}(1) \end{pmatrix}=\begin{bmatrix} 1 & i\epsilon\Omega(\langle\bar{S}_{11}\rangle+\langle\bar{S}_{12}q\rangle) \\ i\epsilon\Omega\langle\bar{R}_{11}\rangle & 1 \end{bmatrix} \times\begin{pmatrix} -\bar{u}(0) \\ \tau_{zz}(0) \end{pmatrix}+O(\epsilon^2). \tag{21}$$

Comparison with the Floquet condition (16) implies that $\bar{k}_0$ is given by

$$\bar{k}_0^2=(\langle\bar{S}_{11}\rangle+\langle\bar{S}_{12}q\rangle)\langle\bar{R}_{11}\rangle. \tag{22}$$

D. The effective medium

It helps to rewrite the fundamental equations (19) and (22) in dimensional form. The latter equation implies that the compressional wave number for waves of the form e^{ikz} is $k=\omega/v^*$, where the complex-valued speed v^* may be written

$$v^*=(C^*/\langle\rho\rangle)^{1/2}. \tag{23}$$

The effective modulus C^* is defined as

$$C^*=\frac{1}{S^*},\quad S^*(\omega)=\left\langle\frac{1-\alpha P}{K+\frac{4}{3}\mu}\right\rangle. \tag{24}$$

The dimensionless pressure $P(z)$ is the solution to the following diffusion problem defined on $0<z<L$,

$$\frac{dP(z)}{dz}=\frac{-\eta}{\kappa}W(z),\quad \frac{dW(z)}{dz}=\frac{i\omega\alpha}{K+\frac{4}{3}\mu}[R^{-1}P(z)-1], \tag{25}$$

subject to the periodic boundary conditions: $P(L)=P(0)$ and $W(L)=W(0)$. The material parameter $R(z)$ is

$$R(z)=\alpha M/(K_c+\tfrac{4}{3}\mu). \tag{26}$$

If $R(z)$ is constant then it is easily checked that $W=0$ and $P=R$, and the effective modulus is simply C_∞ of Eq. (13). In this case the effective one-dimensional modulus is real and equal to the harmonic average of the one-dimensional confined plane strain modulus. In order to obtain a complex-valued C^*, and hence attenuation, R must vary within the unit period. This could result from changes in any or all of the parameters ϕ, α, K, μ, and K_f, but is unaffected by changes in ρ, η, or κ. If the frame is relatively stiff, i.e., $K\gg K_f$, then a good approximation is $M\approx K_f/\phi$, and correspondingly

$$R\approx\frac{\alpha}{\phi}\frac{K_f}{(K+\frac{4}{3}\mu)}. \tag{27}$$

Hence R is small in the stiff frame approximation, but it may suffer large relative change if, for instance, K_f changes drastically. This is the mechanism behind the White model for rocks with partial gas saturation.[9-13] The ratio of K_f for gas to that of either oil or water is on the order of 1:40 for gases under substantial confining pressure or at great depths, and less for unpressurized gas. One would therefore expect the greatest effects for layers consisting of alternating fluid and gas zones, which is precisely the model solved by White *et al.*[10] and discussed in Appendix A, where it is shown that the solution of Eq. (25) with periodic boundary conditions gives their effective modulus exactly.

E. The low- and high-frequency limiting moduli

The definition of the low- and high-frequency regimes depends upon the length of the period L, and upon the spatially dependent diffusion coefficient of Eq. (25),[17,21,22]

$$D(z)=\frac{\kappa}{\eta}M\left(\frac{K+\frac{4}{3}\mu}{K_c+\frac{4}{3}\mu}\right), \tag{28}$$

which simplifies to $D\approx D_0$ (introduced in Sec. I) in the stiff frame limit. The low and high frequency regimes are, respectively, $\omega\ll D/L^2$, and $\omega\gg D/L^2$. In the high frequency limit, the term in square brackets in Eqs. (25) vanishes uniformly, or $P(z)=R(z)$, and it follows from Eqs. (24) and (26) that $C^*\to C_\infty$. In the low frequency or static limit, Eq. $(25)_2$ implies that W=constant, but the periodicity condition $P(L)=P(0)$ combined with Eq. $(25)_1$ requires $W\equiv 0$, and hence P=constant. The constant value follows from averaging $(25)_2$, and when substituted into Eq. (24) yields $C^*\to C_0$, where the low-frequency or quasistatic modulus is

$$\frac{1}{C_0}=\left\langle\frac{1}{K+4/3\mu}\right\rangle-\left\langle\frac{\alpha}{K+4/3\mu}\right\rangle^2\left\langle\frac{\alpha}{R(K+4/3\mu)}\right\rangle^{-1}. \tag{29}$$

Let $\delta\equiv\alpha/(K+4/3\mu)$, then it follows from (4), (13), and (29), that

$$\frac{1}{C_0}-\frac{1}{C_\infty}=\left\langle\frac{\delta}{R}\right\rangle^{1}\left(\langle\delta R\rangle\left\langle\frac{\delta}{R}\right\rangle-\langle\delta\rangle^2\right)>0, \tag{30}$$

with equality if, and only if, R is constant. Hence, the limiting high-frequency speed exceeds the limiting low-frequency speed when R varies. For instance, in a fairly uniform and stiff rock structure with alternating fluid and gas saturation, α, $\delta\approx$constant and $R\ll 1$, and it can be shown from (23) and (30) that

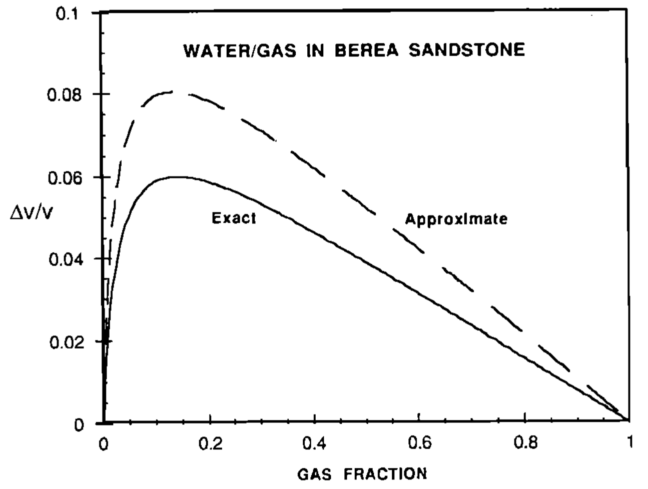

FIG. 1. The magnitude of $\Delta v/v$ for compressional waves in Berea sandstone with periodic layers of gas and water saturation. The fluid parameters are given in Table II. The exact result follows from Eqs. (13), (23), and (29), and the approximate result from Eq. (34).

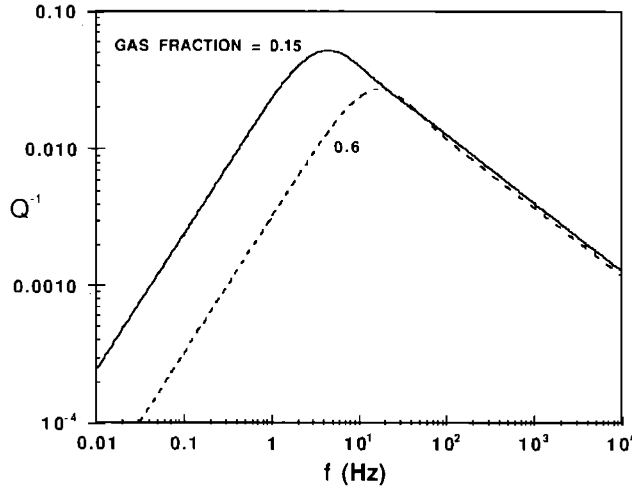

FIG. 2. The quality factor for alternating layers of water/gas in Berea, with unit period of 1 m.

$$\frac{\Delta v}{v} \approx \frac{\alpha}{2}\left(\langle R\rangle - \left\langle \frac{1}{R}\right\rangle^{-1}\right). \tag{31}$$

It is shown in Appendix B that the time-dependent response function associated with $S^*(\omega)$ is causal. This means that the effective stress strain relation can be expressed in terms of a creep function $\hat{S}(t)$ in the form

$$e_{zz}(z,t) = S_\infty \tau_{zz}(z,t) + \hat{S}(\cdot)*\tau_{zz}(z,\cdot)(t), \tag{32}$$

where the * indicates convolution, $S_\infty = 1/C_\infty$,

$$\hat{S}(t) = \frac{1}{\pi} H(t) \mathrm{Re} \int_0^\infty (S^* - S_\infty) e^{-i\omega t}\, d\omega, \tag{33}$$

and $H(t)$ is the Heaviside unit step function. Thus, S_∞ defines the instantaneous response, and $\hat{S}(t)$ represents an additional, generally small, viscoelastic response. The magnitude of the viscoelasticity can be gauged from the identity $\int_0^\infty \hat{S}(t)dt = S_0 - S_\infty \geqslant 0$, where $S_0 = 1/C_0$ [see Eq. (30)]. An example of the creep function is given next.

F. The two-phase model and numerical results

The two constituent layered model of White *et al.*[10] is probably the simplest model from an analytical point of view, with the explicit solution given in Eq. (A3). The specific case of alternating water and gas regions in Berea sandstone is chosen; the fluid parameters are summarized in Table II. The range of dispersion, $\Delta v/v = (v_\infty - v_0)/[(v_\infty + v_0)/2]$, is plotted in Fig. 1 as a function of the volume fraction of pore space occupied by gas. It is clear that maximum dispersion occurs for relatively small concentrations of gas, and would occur at even smaller values if the gas compressibility were greater. The gas considered here has a relatively high bulk modulus, corresponding to large overburden pressure, for example. It is interesting to examine the approximation in Eq. (31) for this case. It may be simplified further by combining it with Eq. (27) to give for a gas/fluid alternating sequence in a stiff frame,

$$\frac{\Delta v}{v} \approx \frac{\alpha^2}{2\phi}\left(K + \frac{4}{3}\mu\right)^{-1} n_g(1-n_g)\, \frac{(K_f - K_g)^2}{n_g K_f + (1-n_g)K_g}, \tag{34}$$

where n_g is the fraction of material infiltrated by gas. It is clear from Fig. 1 that although Eq. (34) overestimates the dispersion by about 30%, it correctly predicts that the greatest dispersion occurs at a gas fraction of about 15%.

The change in wave speed is of secondary importance in comparison to the attenuation associated with the dispersion. This can be described in terms of the dimensionless Q factor, where $1/v^* = (1 + i/2Q)/v$, and $v(\omega)$ is real. This is plotted in Fig. 2 as a function of frequency for $L = 1$ m, for two different values of gas concentration. The frequency dependence was discussed extensively by White *et al.*[10] for the present model and by Dutta and Odé[11,12] for the spherical model, and it generally has the form shown in Fig. 2. These references also provide plots of the wave speed dispersion as a function of frequency; for purposes of brevity the frequency dependence of the speed will not be discussed here. The single peak in $1/Q$ in Fig. 2 is characteristic of a single dominating relaxation process, associated with the oscillatory diffusion of water into the gas, and vice versa (note that in the present case the two diffusion coefficients are of comparable magnitude). In realistic situations one can expect that the peak will be smeared out over a wide frequency range. The peak in $1/Q$ occurs at different frequencies, depending upon the relative concentration of water and gas, and is plotted in Fig. 3. The associated minimum value of Q is plotted in Fig. 4, from which it is evident that the greatest attenuation occurs at roughly the same gas fraction that gives the largest dispersion (see Fig. 1). Note that the dependence of C^* and v^* upon frequency and the periodic length L occurs in the combination (fL^2), as pointed out by Dutta and Odé.[11,12] The numerical results shown here are all for $L = 1$ m, so for example, if $L = 50$ cm, the frequency scales in Figs. 2 and 3 should be multiplied by a factor of 4.

The viscoelastic creep function $\hat{S}(t)$ of Eq. (33) can be

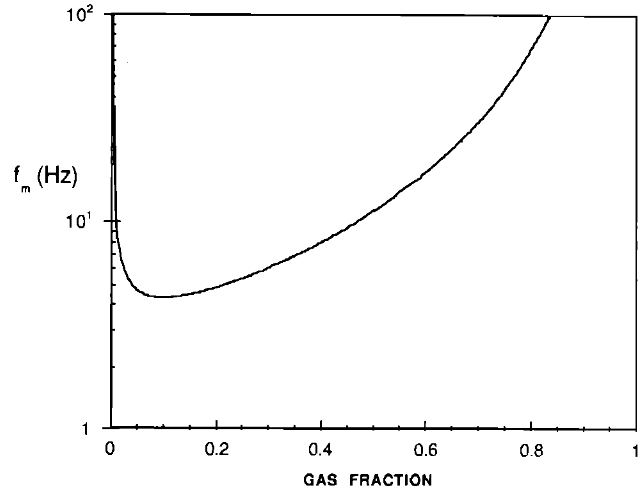

FIG. 3. The frequency at which the quality factor is maximum in gas/water saturated Berea with $L = 1$ m.

computed using the explicit form of S^* in Appendix A. The high frequency form of S^* implies that $\hat{S}(t)$ has an inverse square root singularity for $t \downarrow 0$. The long time response is governed by the location of the singularity of $S^*(\omega)$ which lies closest to the real axis in the ω plane. Denoting this by $\bar{\omega}$, the behavior for large t is exponentially small; thus, $|\hat{S}| < S_\infty \exp(-\bar{\omega} t)$. For the two-phase model of Appendix A, it can be shown that $\bar{\omega} > \pi^2 \min(D_1/L_1^2, D_2/L_2^2)$. The transition from one regime to the other is evident in Fig. 5, and the characteristic time for the transition is clearly related to the frequency at which the attenuation peaks (see Fig. 2).

III. THE LIMITING MODULI

The same macroscopic viscoelastic behavior found in the one-dimensional layered medium is also expected for media which are periodic in all three directions. However, the cases of practical interest are not periodic although they do display macroscopic homogeneity, as in the earth.

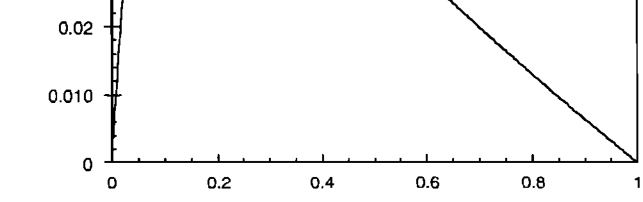

FIG. 4. The minimum possible Q as a function of gas concentration for the gas/water saturated Berea model.

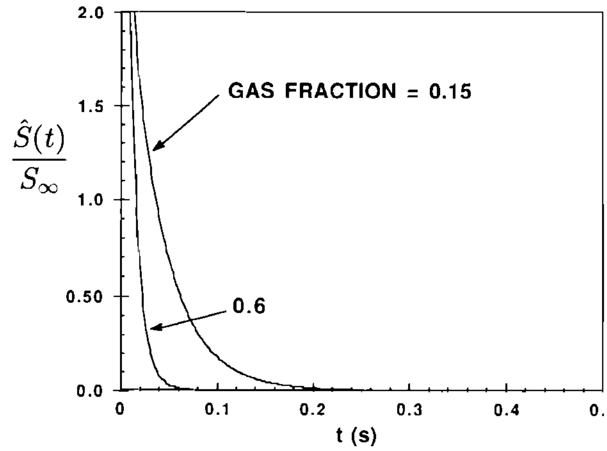

FIG. 5. The creep function of Eq. (33) for the layered model. The inverse FFT was accomplished after first subtracting out the singular part.

The simplest extension of the theory to such media would involve a "representative" unit cell characteristic of the entire medium. This is well defined for the layered periodic medium, but generally requires some statistical assumptions about the microstructure. It is interesting to note that the original theory of White,[9] for a spherical region of gas saturation surrounded by a shell of fluid infiltrated rock, is not directly related to a periodic medium. A realization of the White model would consist, not of periodic arrangements of the unit shell/sphere, but of a space filling composed of shell/spheres of different sizes which fill up all the intervening gaps. This type of model also realizes the Hashin–Shtrikman bounds in the theory of linear static elasticity for two-phase composite materials.[23,24]

Despite the limited number of models for which closed form expressions are available, it is possible nonetheless to make some general statements about the limiting low- and high-frequency moduli that govern elastic wave propagation. A discussion of these moduli for the spherical White model may be found in references.[11,12] In general, no matter how complex the microstructure, the disparity between these limiting moduli indicates the range of dispersion due to diffusion, and if the exact frequency dependence were known the Kramers–Kronig relations could be used to derive the attenuation as a function of frequency. However, it is safe to say that the maximum attenuation can be expected when the dispersion is also the greatest, as borne out by the limited numerical results discussed above. It is also simpler to describe the limiting moduli than the full frequency dependence, and for that reason the remainder of this paper will focus on some general properties of the low- and high-frequency effective elastic moduli of a representative sample, i.e., the analogs of C_0 and C_∞ discussed above. The sample region is a volume of sufficient extent that it contains a statistically representative distribution of variations, whether the overall medium is periodic or not.

The high-frequency, or no-flow, moduli can be defined from the point-wise constitutive relations for an arbitrarily anisotropic frame, which are given by Eq. (3), where $\mathbf{L}_c$ are the anisotropic confined moduli. Equation (3) could be generalized to account for anisotropic pore fluid effects, for instance, if the pores have a preferred direction. We will not include this possibility here. In the high-frequency limit the fluid viscosity restrains any fluid motion, with the result that there is no flow throughout the material, i.e., $\xi \equiv 0$, or from Eq. (3),

$$\tau = \mathbf{L}_c \mathbf{e}. \tag{35}$$

The inhomogeneous medium behaves as a composite elastic continuum with local stiffness $\mathbf{L}_c$, and the bulk or effective medium is governed by the effective elastic moduli of the composite. The evaluation of the effective stiffness tensor is a nontrivial matter, but there is enough literature on the subject that it need not be of concern here; see, for example, Refs. 24, 25 for reviews of the subject. Thus, determination of the high-frequency moduli is, in principle, a well-defined and well-researched problem.

In the low-frequency limit, the pressure p is constant, although not necessarily at the ambient level ($p=0$). Its precise value is governed by the global condition that fluid neither enters nor leaves the sample, i.e.,

$$\langle \xi \rangle = 0, \tag{36}$$

where the angular brackets now denote the spatial average over the representative sample. The constitutive relations (3) then simplify to

$$\tau = \mathbf{L}\mathbf{e} - \alpha p \mathbf{I}, \quad \xi = \alpha e + p/M, \tag{37}$$

where $\mathbf{L}$ are the associated frame moduli [K and μ in the isotropic case, see Eq. (4)]. The prevailing value of pressure follows from (36) and $(37)_2$, and when substituted in $(37)_1$ yields

$$\tau = \mathbf{L}\mathbf{e} + \alpha \langle M^{-1} \rangle^{-1} \langle \alpha e \rangle \mathbf{I}. \tag{38}$$

This modified constitutive relation is used in Appendices C and D to derive the limiting low-frequency moduli for two analytically tractable configurations. In Appendix C the full set of transversely isotropic moduli are obtained for an arbitrarily layered medium consisting of isotropic layers, generalizing the model of White *et al.*[10] and also generalizing the purely elastic theory of Backus.[26] The spherically symmetric two-phase model of White[9] is discussed in Appendix D. In both cases, explicit formulae are derived for the low- and high-frequency limiting moduli, based upon the stress-strain relations Eqs. (38) and (35), respectively.

A. Thermoelastic analogy

An alternative approach may be used to find the limiting moduli, based upon the correspondence between the static theory of poroelasticity and the static theory of thermoelasticity which includes entropy. The correspondence was often discussed by Biot in his works, e.g., Ref. 27, and has recently been used to advantage in finding some of the effective parameters in two-phase poroelastic media.[28,29] The correspondence is summarized in Appendix E, from which it is clear that the frame moduli $\mathbf{L}$ in the porous medium correspond to the isothermal moduli of a thermoelastic medium. The connection between the other parameters is summarized in Eq. (E1). The correspondence is very useful because there already exists a fairly large literature on estimating the effective parameters of inhomogeneous thermoelastic continua, e.g., Refs. 25, 24, 30. The problem of finding the effective moduli in the porous medium is therefore related to that of finding the effective modulus, the effective tensor of thermal expansion, and the effective heat capacities for an inhomogeneous thermoelastic medium.

In the case of the high-frequency moduli, the constraint that $\xi=0$ everywhere corresponds to zero entropy change in the thermoelastic medium, and the corresponding effective moduli are based upon the isentropic moduli of the inhomogeneous medium. However, the thermoelastic analogy is not directly useful for the determination of the high-frequency moduli of the porous medium, since the issue at stake is essentially the same in either case. On the other hand, the thermoelastic analogy is very helpful in determining the low-frequency moduli because there is an extensive literature on the corresponding problem in thermoelasticity. Consider, for instance, the low-frequency effective bulk modulus in a macroscopically isotropic medium subject to the constraint $\xi^* \equiv \langle \xi \rangle = 0$, from Eq. (36). The appropriate modulus is K_c^*, which follows from relation $(4)_1$ for the effective medium as $K_c^* = K^* + (\alpha^*)^2 M^*$, where α^* and M^* pertain to the effective medium. It is worth noting that Brown and Korringa[31] introduced new microstructural moduli analogous to K_g and K_f, which are essentially equivalent to α^* and M^*. In terms of the thermoelastic analogy, the effective frame modulus K^* corresponds to the isothermal modulus, and α^* and M^* are related to the effective thermal expansion coefficient and the effective heat capacity of the thermoelastic medium. Things simplify considerably for the case of a two-phase composite medium which is macroscopically isotropic, to the extent that explicit formulas are given in Eq. (E2) for the effective parameters α^* and M^* in terms of the frame bulk modulus K^*, and the constituent values of α and M in the two phases. The derivation of Eq. (E2) follows directly from the thermoelastic analogy and known results in the literature[24] for two-phase media. Combining (E2) and the identity $(4)_1$ for the effective material implies that the low-frequency modulus in any isotropic two-phase medium is

$$K_c^* = K^* + \frac{[\langle \alpha \rangle (K_1 - K_2) - (\langle K \rangle - K^*)(\alpha_1 - \alpha_2)]^2}{\langle M^{-1} \rangle (K_1 - K_2)^2 + (\langle K \rangle - K^*)(\alpha_1 - \alpha_2)^2}. \tag{39}$$

For example, in the spherical White model, discussed in Appendix D, the effective modulus K^* can be found quite easily,[24]

$$K^* = \frac{K_1 K_2 + \frac{4}{3}\mu_2 \langle K \rangle}{K_1 K_2 \langle 1/K \rangle + \frac{4}{3}\mu_2}. \tag{40}$$

Combining Eqs. (39) and (40) shows that K_c^* is identical to K_0 of Eq. (D3), which was derived using the poroelasticity equations. This single example demonstrates the utility of the thermoelastic analogy, which could be used for more general microgeometries.

IV. CONCLUSION

Wave propagation in a spatially inhomogeneous fluid-saturated medium has been analyzed from first principles by a top down approach, i.e., looking at it on the macroscopic level of the wavelength, applying all of the necessary equations of motion and then making appropriate approximations based upon the small parameters in the problem. The "macroscopic" effective medium is viscoelastic with a well-defined, complex valued, frequency-dependent, elastic modulus, which may be obtained by solving an oscillatory diffusion problem on the scale of the inhomogeneity. The specific example of a periodically layered medium was considered in detail, and it has been shown that the effective stiffness for a two-phase medium is in exact agreement with that of White *et al.*[10] The theoretical analysis also implies that if the scaling of Eq. (2) applies then there is no need to analyze the diffusion problem using the more sophisticated Biot theory on the small scale. The "microproblem" can be adequately dealt with in terms of the usual quasistatic theory of diffusion in rocks.[5,22]

Although the solution of the oscillating diffusion problem is generally a nontrivial matter, some idea of the level of dispersion can be gained from knowledge of the limiting low- and high-frequency moduli. These are easier to determine and can be related to two distinct static problems for the inhomogeneous poroelastic medium, each associated with a different physical constraint. Explicit expressions have been obtained for the limiting bulk moduli for the spherical White[9] model (Appendix D), and for the full set of transversely isotropic moduli for a layered medium (Appendix C). The general problem of determining these moduli can be related to the issue of finding the effective elastic and thermal parameters in an inhomogeneous thermoelastic material. It has been shown that by using this correspondence the limiting low-frequency bulk modulus for the spherical shell model of White[9] can be obtained in a relatively simple manner.

ACKNOWLEDGMENTS

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APPENDIX A: A TWO-PHASE STRATIFIED MEDIUM

Consider alternating uniform layers of medium 1 and 2, of lengths L_1 and L_2, $L_1+L_2=L$. Equations (25) can be solved within each layer as follows,

$$\begin{aligned} P_j &= A_j \cos(d_j x/L_j) + B_j \sin(d_j x/L_j) + R_j, \\ W_j &= (1/Z_j) A_j \sin(d_j x/L_j) \\ &\quad - (1/Z_j) B_j \cos(d_j x/L_j), \end{aligned} \tag{A1}$$

where the suffix $j=1$ or 2 indicates the layer, and

$$Z = \frac{\eta}{\kappa}\left(\frac{D}{\omega}\right)^{1/2} e^{-i\pi/4}, \quad d = \left(\frac{\omega}{D}\right)^{1/2} L e^{i\pi/4}. \tag{A2}$$

The effective modulus follows from Eqs. (24) and (A1) as

$$\frac{1}{C^*} = \left\langle \frac{1}{K_c + 4/3\mu} \right\rangle - \sum_{j=1,2} \frac{\alpha_j}{(K+4/3\mu)_j} \frac{L_j}{d_j L} \times [A_j \sin d_j + (-1)^j B_j (1 - \cos d_j)]. \tag{A3}$$

The coefficients A_j, B_j, $j=1,2$ are determined from the four conditions at the interfaces that P and W must be continuous and periodic in L. Omitting the details, it can be shown that

$$\frac{1}{C^*} = \frac{1}{C_\infty} + \frac{2}{i\omega L}(R_1 - R_2)^2 \left(Z_1 \cot\frac{d_1}{2} + Z_2 \cot\frac{d_2}{2} \right)^{-1}, \tag{A4}$$

which is identical to the result of White *et al.*,[10] derived by a slightly different approach. It follows from the explicit form of (A3) that the low- and high-frequency limits of C^* are $C_0 + O(\omega)$ and $C_\infty + O(\omega^{-1/2})$, respectively.

APPENDIX B: ANALYTIC PROPERTIES OF THE EFFECTIVE MODULUS

Equations (25) are a pair of coupled forced equations for which the corresponding homogeneous system is

$$\frac{dP(z)}{dz} = -a_1(z) W(z), \quad \frac{dW(z)}{dz} = i\omega a_2(z) P(z), \tag{B1}$$

where $a_1 = \eta/\kappa$ and $a_2 = M^{-1}(K_c + 4\mu/3)/(K + 4\mu/3)$. Multiply $(\text{B1})_2$ by the conjugate of P, integrate by parts and use $(\text{B1})_1$ and the periodicity conditions on P and W to deduce that

$$\omega = -i \frac{\int_0^L a_2 |P|^2 \, dz}{\int_0^L a_1 |W|^2 \, dz}. \tag{B2}$$

Because a_1 and a_2 are non-negative, any solution to the homogeneous equations must correspond to a frequency on the negative imaginary axis. Any singularities (blow-ups) of the solution to Eqs. (25) must be associated with nontrivial solutions to the homogeneous equations, and hence can occur only for values of frequency on the negative imaginary axis. These may be discrete or continuous, corresponding to poles and branch cuts, but their precise distribution is immaterial to the present argument. By implication, singularities of the effective compliance $S^*(\omega)$ of Eq. (24) can only lie on the negative imaginary axis in the complex ω plane. The time-dependent response function associated with $S^*(\omega)$ is therefore real and causal by virtue of the properties that $S^*(\omega)$ is analytic in the upper half-plane and $S^*(-\omega) = \text{c.c. } S^*(\omega)$ for ω real.

APPENDIX C: HIGH- AND LOW-FREQUENCY MODULI OF A LAYERED MEDIUM

We consider here a medium composed of layered isotropic Biot constituents, with the layering in the x_3 direction. The high-frequency or no-flow moduli follow by applying the theory of Backus[26] to (3) with $\xi \equiv 0$, resulting in an effective medium with transverse isotropy and moduli C_{11}, C_{13}, C_{33}, C_{55}, and C_{66}. These are obtained from the

Backus algorithm using the inhomogeneous moduli K_c and μ. In particular, $C_{33} \equiv C_\infty$, where C_∞ is defined in Eq. (13).

The low-frequency moduli $\bar{C}_{11}$, $\bar{C}_{13}$, $\bar{C}_{33}$, $\bar{C}_{55}$, and $\bar{C}_{66}$ are defined by the constraint that p is constant such that $\langle \xi \rangle = 0$. The effective shear moduli are unaffected, so that $\bar{C}_{55} = C_{55}$, $\bar{C}_{66} = C_{66}$. The effective moduli of a layered anisotropic medium can be generated by rewriting the stress-strain relations in a form that expresses the variables $(e_{13}, e_{23}, e_{33}, \tau_{11}, \tau_{22}, \tau_{12})$ in terms of the global constants $(e_{11}, e_{22}, e_{12}, \tau_{13}, \tau_{23}, \tau_{33})$. The same procedure is used here, with the distinction that the apparent starting point, Eq. (38), already contains an average of e_{33}. We can eliminate this by first using the general relations to express ξ in terms of "constants," yielding

$$\xi = 2\mu \frac{\alpha}{\gamma}(e_{11}+e_{22}) + \frac{\alpha}{\gamma}\tau_{33} + \left(\frac{1}{M} + \frac{\alpha^2}{\gamma}\right)p, \tag{C1}$$

where $\gamma = \lambda + 2\mu$. The condition $\langle \xi \rangle = 0$ implies

$$p = -A(e_{11}+e_{22}) - B\tau_{33}, \tag{C2}$$

where

$$A = \left\langle \frac{1}{M} + \frac{\alpha^2}{\gamma} \right\rangle^{-1} \left\langle 2\mu \frac{\alpha}{\gamma} \right\rangle, \quad B = \left\langle \frac{1}{M} + \frac{\alpha^2}{\gamma} \right\rangle^{-1} \left\langle \frac{\alpha}{\gamma} \right\rangle. \tag{C3}$$

Substitution of (C2) into $(37)_1$ gives, ignoring shear stresses and strains,

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ e_{33} \end{bmatrix} = \frac{1}{\lambda + 2\mu} \begin{bmatrix} 2\mu(2\lambda + 2\mu + \alpha A) & 2\mu(\lambda + \alpha A) & \lambda + 2\mu\alpha B \\ 2\mu(\lambda + \alpha A) & 2\mu(2\lambda + 2\mu + \alpha A) & \lambda + 2\mu\alpha B \\ -(\lambda + \alpha A) & -(\lambda + \alpha A) & 1 - \alpha B \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ \tau_{33} \end{bmatrix}. \tag{C4}$$

The average values of the quantities on the left of this equation follow by averaging the matrix on the right, since e_{11}, e_{22}, and τ_{33} are constant, by assumption. The effective moduli can then be read off, yielding

$$\bar{C}_{33} = \left\langle \frac{1 - \alpha B}{\lambda + 2\mu} \right\rangle^{-1}, \quad \bar{C}_{13} = \left\langle \frac{\lambda + \alpha A}{\lambda + 2\mu} \right\rangle \bar{C}_{33},$$
$$\bar{C}_{11} = \frac{\bar{C}_{13}^2}{\bar{C}_{33}} + \langle 2\mu \rangle + \left\langle 2\mu \left(\frac{\lambda + \alpha A}{\lambda + 2\mu} \right) \right\rangle. \tag{C5}$$

We note that $\bar{C}_{33} \equiv C_0$, defined in Eq. (29).

It is interesting to note that in each of the low-frequency moduli $\bar{C}_{11}$, $\bar{C}_{13}$, and $\bar{C}_{33}$, the effect of the pore fluid enters only through the parameter $\langle \alpha^2/(\lambda+2\mu) + 1/M \rangle^{-1}$, in which M enters only through its harmonic average. In the stiff frame approximation this term is approximately $\langle \phi/K_f \rangle^{-1}$, which is much smaller than terms like $\langle 1/(\lambda+2\mu) \rangle^{-1}$. The differences between the low-frequency moduli and the no-flow moduli are therefore small and on the order of $\langle \phi/K_f \rangle^{-1}$ in magnitude. The effective moduli can be simply approximated in this limit. For example, if the frame is spatially uniform but the parameter M varies, then the stiff-frame approximation implies that to first order in M the low-frequency effective medium is isotropic with shear modulus μ and bulk modulus

$$\bar{K} = K + \alpha^2 \langle M^{-1} \rangle^{-1}. \tag{C6}$$

However, the no-flow or high-frequency moduli are transversely isotropic to first order with $C_{55} = C_{66} = \mu$ and

$$C_{11} = \lambda + 2\mu + \left(\frac{2\mu - \lambda}{2\mu + \lambda} \right) \alpha^2 \langle M \rangle,$$
$$C_{33} = \lambda + 2\mu + \alpha^2 \langle M \rangle, \quad C_{13} = \lambda. \tag{C7}$$

APPENDIX D: THE LIMITING MODULI FOR WHITE'S SPHERICAL MODEL

White's[9] model of a gas pocket consists of a spherical region of radius a with material 1 (gas) surrounded by a shell of material 2 with exterior radius b. Consider the low-frequency limiting moduli first. Let e_1 and e_2 be the averaged bulk dilatation in each region, then the spherically symmetric solution to the static equations of elasticity with Eq. (38) yields

$$u_r = \begin{cases} \frac{e_1}{3} r, & r < a, \\ \frac{e_2}{3} r - \frac{A}{4\mu_2 r^2}, & r > a, \end{cases}$$
$$\tau_{rr} = \begin{cases} K_1 e_1 + \alpha_1 B, & r < a, \\ K_2 e_2 + A/r^3 + \alpha_2 B, & r > a, \end{cases} \tag{D1}$$

where A and B are constants with $B = \langle M^{-1} \rangle^{-1} \langle \alpha e \rangle$. The displacement at the exterior surface $(r=b)$ is related to the average strain by $u_r = \langle e \rangle b/3$. The effective low-frequency modulus is then $K_0 = \tau_{rr}(b)/\langle e \rangle$, or by simple manipulations,

$$K_0 = (\langle Ke \rangle + B\langle \alpha \rangle)/\langle e \rangle. \tag{D2}$$

The unknowns e_1, e_2, and A are found using the two conditions that u_r and τ_{rr} are continuous at $r=a$. Substitution of e_1 and e_2 into Eq. (D2) gives

$$K_0 = \frac{K_1 K_2 + \frac{4}{3}\mu_2 \langle K \rangle + \langle M^{-1} \rangle^{-1} (K_1 K_2 \langle \alpha^2/K \rangle + \frac{4}{3}\mu_2 \langle \alpha \rangle^2)}{K_1 K_2 \langle 1/K \rangle + \frac{4}{3}\mu_2 + \langle M^{-1} \rangle^{-1} (\langle \alpha^2 \rangle - \langle \alpha \rangle^2)}. \tag{D3}$$

Dutta and Odé[12] obtained K_0 in the particular case that the frame is the same in both regions, i.e., $K_1=K_2=K$, $\alpha_1=\alpha_2=\alpha$, and $\mu_1=\mu_2=\mu$, for which Eq. (D3) simplifies to

$$K_0=K+\alpha^2\langle M^{-1}\rangle^{-1}, \tag{D4}$$

in agreement with Eq. (A-27) of Dutta and Odé.[12] Note the similarity with the approximation Eq. (C6). Equation (D4) can be rewritten $K_0=K+\alpha^2\bar{M}$, where $\bar{M}$ is given by Eq. $(9)_2$ but with K_f replaced by its harmonic average, i.e., Wood's formula.

The high-frequency or no-flow bulk modulus follows from Eq. (D3) by deleting the terms involving $\langle M^{-1}\rangle^{-1}$ and replacing K_1 and K_2 by the corresponding confined moduli,

$$K_\infty=\frac{K_{c1}K_{c2}+\frac{4}{3}\mu_2\langle K_c\rangle}{K_{c1}K_{c2}\langle 1/K_c\rangle+\frac{4}{3}\mu_2}. \tag{D5}$$

If $(K_{c1}-K_{c2})(\mu_1-\mu_2)>0$, then Eq. (D5) is precisely the Hashin–Shtrikman upper (lower) bound[23,25] for the modulus of any isotropic composite medium of materials 1 and 2 such that $\mu_2>\mu_1(\mu_2<\mu_1)$. If $\mu_1=\mu_2$, the bounds coincide and thus Eq. (D5) is exact, independent of the microstructure, a result due to Hill.[32] This means that Eq. (D5) is exact for any uniform frame (i.e., $K_1=K_2$, $\mu_1=\mu_2$ and $\alpha_1=\alpha_2$) infiltrated by two fluids in a statistically isotropic manner, in which case the modulus simplifies to

$$K_\infty=K+\alpha^2\left(\frac{(K+\frac{4}{3}\mu)\langle M\rangle+\alpha^2M_1M_2}{K+\frac{4}{3}\mu+\alpha^2M_1M_2\langle 1/M\rangle}\right). \tag{D6}$$

The difference between the high- and low-frequency moduli is therefore, from Eqs. (D4) and (D6),

$$K_\infty-K_0=[\alpha^{-2}+M_1M_2\langle 1/M\rangle(K+\tfrac{4}{3}\mu)^{-1}]^{-1}\times(\langle M\rangle-\langle 1/M\rangle^{-1})\geqslant 0, \tag{D7}$$

which may be compared with the analogous inequality (30) for the 1-D moduli.

APPENDIX E: THE POROELASTIC-THERMOELASTIC CORRESPONDENCE

The constitutive equations for an isotropic thermoelastic medium[33] can be cast in exactly the same form as Eq. (3) if the following equivalence is observed between the parameters

$$(p,\xi,K_c,\alpha,M)\leftrightarrow(\theta,s,K_s,3K\beta,1/C_v). \tag{E1}$$

Here, θ is the temperature variation from its ambient value θ_0, s is the entropy per unit volume, β is the coefficient of thermal expansion, and θ_0C_v is the heat capacity per unit volume at constant stress. The isentropic bulk modulus K_s is related to the isothermal modulus K by a formula analogous to $(4)_1$. Having made the correspondence Eq. (E1), known results for the effective values of β^* and C_v^* in macroscopically isotropic two-component media can be used to get expressions for α^* and M^*. Explicit formulas for β^* and C_p^* are given by Christensen,[24] where $C_p=C_v+9K\beta^2$ is the specific heat at constant pressure. Translating these into the poroelastic variables implies the identities[28,29]

$$\alpha^*-\langle\alpha\rangle=\left(\frac{\alpha_1-\alpha_2}{K_1-K_2}\right)(K^*-\langle K\rangle),$$
$$\frac{1}{M^*}-\left\langle\frac{1}{M}\right\rangle=-\left(\frac{\alpha_1-\alpha_2}{K_1-K_2}\right)^2(K^*-\langle K\rangle). \tag{E2}$$

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